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Sigma Model for Ernst Equation: Lax Representation, Bäcklund Transformation and Divergence-Free Currents

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A sigma model associated with the Ernst equation is derived. This sigma model is described by the Belinsky–Zakharov-type completely integrable equation and is formally equivalent to the usual sigma model in curved two-dimensional space. The corresponding Lax representation, Bäcklund transformation, and divergence-free currents are obtained.

1. INTRODUCTION

The analysis of geometrical structures associated with completely integrable nonlinear evolution equations (Bullough and Caudrey, 1980) has lead to the discovery of interrelations between both different techniques for solving such equations and different equations of this class (Wahlquist and Estabrook, 1975; Hermann, 1976; Barbashov and Nesterenko, 1980). Zakharov and Takhtadjan (1979), bearing in mind the fiber bundle interpretation of the inverse scattering method, introduced a notion of gauge equivalence of nonlinear equations. Namely, two systems of nonlinear equations solvable via the inverse scattering method are gauge equivalent if corresponding connections are defined on the same fiber bundle and are related one with the other by means of a gauge transformation independent of a spectral parameter. In the framework of this approach they established the gauge equivalence of the nonlinear Schrödinger equation and the continuous isotropic Heisenberg spin chain equation. Lakshmanan and Bullough (1980) extended the results of Zakharov and Takhtadjan to the case of the generalized Schrödinger equation (Calogero and Degasperis, 1978). The gauge equivalent obtained represents a respective generalization of the Heisenberg ferromagnet model.

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On the other hand, Pohlmeyer (1976) showed by the reduction procedure that the $O(n)$ invariant nonlinear sigma model is intimately related to the sine-Gordon equation ($n = 3$) or its generalizations. This problem was further considered by Pohlmeyer and Rehren (1979) and Honerkamp (1981). Orfanidis (1980) developed a systematic method of obtaining gauge equivalents for completely integrable nonlinear equations reproducing along these lines, in particular, the results of Zakharov-Takhtadjan and Pohlmeyer. Gauge equivalents derived in such a manner have been called sigma models associated with given nonlinear equations. The similar problem was considered by Reiter (1980).

In the present paper we derive a sigma model associated with the Ernst equation (Ernst, 1968) generalizing thereby the Orfanidis' approach to equations with explicit space dependence. It turns out that such a dependence does not preclude obtaining the corresponding results in a closed form.

Our paper is organized as follows. In Section 2 we briefly review the Ernst equation and give the Lax representation for it. In Section 3 an associated sigma model is obtained. Bäcklund transformations for this sigma model derived in Section 4 can be considered a new type of those for the gravitational field with axial symmetry. Section 5 is devoted to finding divergence-free currents.

2. ERNST EQUATION

The Ernst equation describes axially symmetric, stationary configurations of gravitational field, as well as being equivalent (Witten, 1979; Forgàcs et al., 1980, 1981) to the axially and mirror symmetric Bogomolny equations (Bogomolny, 1976). The most general line element of such configurations can be written

$$ds^2 = f(dt - \omega d\varphi)^2 - \frac{1}{f}\left[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2\right]$$

where functions f, ω, and γ depend on cylindrical coordinates ρ and z and are independent of t and φ. The vacuum Einstein equations relevant to this metric are divided into two parts, of which one includes equations for only f and ω

$$\Delta \ln f + \frac{f^2}{\rho^2}(\nabla\omega)^2 = 0 \tag{1a}$$

$$\frac{\rho}{f}\Delta\omega - 2\nabla\omega\nabla\frac{\rho}{f} = 0 \tag{1b}$$

with $\nabla = (\partial_z, \partial_\rho)$, $\Delta = \partial_\rho^2 + \rho^{-1}\partial_\rho + \partial_z^2$, while the other part gives the function γ in terms of f and ω. Usually the integration of the equations for γ with known f and ω does not lead to difficulties, by virtue of which we shall concentrate in the sequel on the equations (1). Define a new function $\tilde{\psi}$ by means of the relations

$$\tilde{\psi}_\rho = \frac{f^2}{\rho}\omega_z, \qquad \tilde{\psi}_z = -\frac{f^2}{\rho}\omega_\rho$$

where subscripts stand for partial derivatives. Equation (1b) can be then considered as the integrability condition for $\tilde{\psi}$. Introducing the complex Ernst potential $\varepsilon = f + i\tilde{\psi}$, equations (1) take the compact form

$$(\mathrm{Re}\,\varepsilon)\Delta\varepsilon - (\nabla\varepsilon)^2 = 0 \tag{2}$$

Equation (2) is called the Ernst equation. In the coordinates $\xi = \rho + iz$ and $\eta = \rho - iz$ the Ernst equation is represented as follows:

$$\varepsilon_{\xi\eta} - \frac{2\varepsilon_\xi\varepsilon_\eta}{\varepsilon + \varepsilon^*} + \frac{1}{2}\frac{\varepsilon_\xi + \varepsilon_\eta}{\xi + \eta} = 0 \tag{3}$$

where the asterisk denotes complex conjugation. Just this form of the Ernst equation will be used in what follows.

As was pointed out earlier (Maison, 1978; Belinsky and Zakharov, 1978, 1979; Hauser and Ernst, 1980; Chinea, 1981) the Ernst equation is solvable by the inverse scattering method. In particular, it admits the Lax representation (the zero-curvature representation)

$$\psi_\xi = U\psi, \qquad \psi_\eta = V\psi \tag{4}$$

where matrices U and V depend on ε, ε_ξ, ε_η, and their complex conjugate, as well as on ξ, η, and a spectral parameter λ. The compatibility condition for the system (4)

$$U_\eta - V_\xi + [U, V] = 0 \tag{5}$$

must coincide with the Ernst equation. Our Lax representation differs from known ones and has the form

$$\begin{aligned} U &= \sigma(M_1 Z_+ + M_2 Z_-) + \tfrac{1}{2}(M_1 - M_2)Z_3 \\ V &= \sigma^{-1}(N_1 Z_+ + N_2 Z_-) + \tfrac{1}{2}(N_1 - N_2)Z_3 \end{aligned} \tag{6}$$

with

$$M_1 = \frac{\varepsilon_\xi}{\varepsilon + \varepsilon^*}, \qquad M_2 = \frac{\varepsilon_\xi^*}{\varepsilon + \varepsilon^*}, \qquad N_1 = \frac{\varepsilon_\eta}{\varepsilon + \varepsilon^*}, \qquad N_2 = \frac{\varepsilon_\eta^*}{\varepsilon + \varepsilon^*},$$

$$\sigma^2 = \frac{\lambda - i\eta}{\lambda + i\xi}$$

The generators $Z_\pm$ and Z_3 which obey the commutation relations $[Z_3, Z_\pm] = \pm 2Z_\pm$, $[Z_+, Z_-] = Z_3$ are expressed in terms of the $su(1,1)$ algebra basis τ_i:

$$Z_\pm = \tfrac{1}{2}(i\tau_1 \mp \tau_2), \qquad Z_3 = \tau_3$$

Here $\tau_1 = -i\sigma_1$, $\tau_2 = -i\sigma_2$, $\tau_3 = \sigma_3$, and σ_i are Pauli matrices.

From the geometrical point of view, matrices U and V can be treated (Hermann, 1976; Doktorov, 1980) as coefficients of a connection form on the associated vector bundle with the base space $\mathbb{R}^2$ and the standard fiber in which a linear representation of the $SU(1,1)$ group acts. Then the system (4) of linear equations represents the parallel transport equations for a given connection, while the condition (5) states that this connection is flat, i.e., a curvature is zero. The changing of a trivialization of the principal fiber bundle induces the corresponding changing of the associated bundle trivialization which is interpreted as a gauge transformation of sections ψ of the associated bundle. (For a comprehensive review of fiber bundles in the physical content see, for instance, Trautman, 1981.)

3. ASSOCIATED SIGMA MODEL

Following Zakharov and Takhtadjan (1979) we shall call two nonlinear equations to be gauge equivalent if the connections U, V and U', V' are defined on the same bundle and are related one with the other via a gauge transformation independent of the spectral parameter λ:

$$U' = gUg^{-1} + g_\xi g^{-1}, \qquad V' = gVg^{-1} + g_\eta g^{-1} \tag{7}$$

At the same time, the sections ψ and ψ' are connected by $\psi' = g\psi$. Here $g(\xi, \eta)$ is an element of the $SU(1,1)$ group representation.

For obtaining a sigma model associated with the Ernst equation we take $g(\xi, \eta) = \psi(\xi, \eta, \lambda = 0)$, where $\psi(\xi, \eta, \lambda)$ is a solution of the linear equations (4). The function $g(\xi, \eta)$ obeys the equations

$$g_\xi = -gU(\lambda = 0), \qquad g_\eta = -gV(\lambda = 0) \tag{8}$$

The new connection coefficients U' and V' are written then in the form

$$U' = g[U - U(\lambda = 0)]\,g^{-1}, \qquad V' = g[V - V(\lambda = 0)]\,g^{-1} \tag{9}$$

We introduce now elements of the moving trihedral (Orfanidis, 1980)

$$E_{\pm} = \tfrac{1}{2}(iE_1 \mp E_2) = gZ_{\pm}g^{-1}, \qquad Q \equiv E_3 = gZ_3g^{-1}, \qquad E_{\pm}^2 = 0, \qquad Q^2 = 1 \tag{10}$$

which obviously satisfy the same commutation relations as $Z_{\pm}$ and Z_3. Just Q will be further identified with a sigma field under consideration. From the definitions (10) and equations (9) we get equations of motion for Q

$$Q_{\xi} = [Q, gU(\lambda = 0)g^{-1}], \qquad Q_{\eta} = [Q, gV(\lambda = 0)g^{-1}]$$

and similar relations for E, which can be written as follows:

$$\begin{aligned}
&Q_{\xi} = 2i(\eta/\xi)^{1/2}(M_1E_+ - M_2E_+), \qquad Q_{\eta} = -2i(\xi/\eta)^{1/2}(N_1E_+ - N_2E_-)\\
&E_{+\xi} = i(\eta/\xi)^{1/2}M_2Q - (M_1 - M_2)E_+,\\
&E_{+\eta} = -i(\xi/\eta)^{1/2}N_2Q - (N_1 - N_2)E_+\\
&E_{-\xi} = -i(\eta/\xi)^{1/2}M_1Q + (M_1 - M_2)E_-,\\
&E_{-\eta} = i(\xi/\eta)^{1/2}N_1Q + (N_1 - N_2)E_-
\end{aligned} \tag{11}$$

This gives the equation for Q of the form

$$Q_{\xi\eta} = 2(M_1N_2 + M_2N_1)\hat{Q} + i\frac{(\xi\eta)^{-1/2}}{\xi + \eta}[(\xi M_1 - \eta N_1)E_+ - (\xi M_2 - \eta N_2)E_-] \tag{12}$$

We must now eliminate from (12) operators $E_{\pm}$ and functions M_i, N_i in favor of Q and its ξ and η derivatives. So, we might show using (11) that the first term on the right-hand side of (12) can be transformed as

$$2(M_1N_2 + M_2N_1)Q = -\tfrac{1}{2}\{Q_{\xi}, Q_{\eta}\}Q$$

where { , } stands for anticommutator. We have further

$$M_1 E_+ = -\frac{i}{8}(\xi/\eta)^{1/2}\big(2Q_\xi + [Q, Q_\xi]\big),$$

$$M_2 E_- = \frac{i}{8}(\xi/\eta)^{1/2}\big(2Q_\xi - [Q, Q_\xi]\big)$$

$$N_1 E_+ = \frac{i}{8}(\eta/\xi)^{1/2}\big(2Q_\eta + [Q, Q_\eta]\big),$$

$$N_2 E_- = -\frac{i}{8}(\eta/\xi)^{1/2}\big(2Q_\eta - [Q, Q_\eta]\big)$$

Inserting these expressions into (12) yields the gauge equivalent for the Ernst equation we need, i.e., associated sigma model

$$2Q_{\xi\eta} + \{Q_\xi, Q_\eta\}Q - \frac{1}{\xi+\eta}\left(\frac{\xi}{\eta}Q_\xi + \frac{\eta}{\xi}Q_\eta\right) = 0, \qquad \det Q = -1 \quad (13)$$

Equation (13) represents a nonlinear relativistic equation which is obviously solvable by the inverse scattering method. In particular, the Lax representation $\psi'_\xi = U'\psi'$, $\psi'_\eta = V'\psi'$ is defined by the matrices

$$U' = \tfrac{1}{2}\big[1 + i\sigma(\xi/\eta)^{1/2}\big]Q_\xi Q, \qquad V' = \tfrac{1}{2}\big[1 - i\sigma^{-1}(\eta/\xi)^{1/2}\big]Q_\eta Q \quad (14)$$

Equation (13) belongs to the Belinsky–Zakharov-type equations (Belinsky and Zakharov, 1978)

$$\big(\alpha Q_\eta Q^{-1}\big)_\xi + \big(\alpha Q_\xi Q^{-1}\big)_\eta = 0, \qquad \alpha_{\xi\eta} = 0 \quad (15)$$

with a particular realization of $\alpha = \xi^{-1} + \eta^{-1}$. It is well known (see, e.g., Horváth and Kiss-Toth, 1982) that the system (15) is invariant under the coordinate transformation of the form $\tilde{\xi} = a(\xi)$, $\tilde{\eta} = a^*(\eta)$, where a is an analytic function. By virtue of this (15) can be formally written as a SU(1,1) sigma model in curved two-dimensional space (Bais and Sasaki, 1982; Mikhailov and Yaremchuk, 1982),

$$\tilde{\nabla}\big[\tilde{\rho}(\tilde{\nabla}Q)Q^{-1}\big] = 0$$

where $\tilde{\nabla} = (\partial_{\tilde{z}}, \partial_{\tilde{\rho}})$, $2\tilde{\rho} = \tilde{\xi} + \tilde{\eta}$, $2\tilde{z} = i(\tilde{\eta} - \tilde{\xi})$, with the Lax representation

$$v_{\tilde{\xi}} = \frac{1}{2}(1 + \tilde{\gamma}^{1/2})Q_{\tilde{\xi}}Q^{-1}v, \qquad v_{\tilde{\eta}} = \frac{1}{2}(1 + \tilde{\gamma}^{-1/2})Q_{\tilde{\eta}}Q^{-1}v, \qquad \tilde{\gamma} = \frac{1 - 2is\tilde{\eta}}{1 + 2is\tilde{\xi}}$$

s is a complex spectral parameter. In this case, however, new coordinates $\tilde{\rho}$ and $\tilde{z}$ lose their "cylindrical" sense.

The vector version of equation (13) was considered recently by Chinea (1983). In the following section we shall derive a Bäcklund transformation for equation (13) which can be treated as new type of matrix Bäcklund transformations for the gravitational field with axial symmetry.

4. BÄCKLUND TRANSFORMATION FOR ASSOCIATED SIGMA MODEL

Let ε be some solution of the Ernst equation and $U(\varepsilon, \lambda), V(\varepsilon, \lambda)$ be the corresponding connection form coefficients. Suppose now there is another solution $\hat{\varepsilon}$ of the Ernst equation with the connection $\hat{U}(\hat{\varepsilon}, \lambda), \hat{V}(\hat{\varepsilon}, \lambda)$ of the same functional form as U and V. Then, by virtue of the gauge invariance of the zero-curvature condition (5), the connection $\hat{U}, \hat{V}$ is related with the connection U, V by means of a gauge transformation

$$\hat{U} = SUS^{-1} + S_\xi S^{-1}, \qquad \hat{V} = SVS^{-1} + S_\eta S^{-1} \tag{16}$$

where $S \in SU(1,1)$ provided ε and $\hat{\varepsilon}$ are related by the Bäcklund transformation (BT). Hence, BT can be realized as gauge transformation of the Lax representation. Such an approach to BT was used by Neveu and Papanicolaou (1978) for the sine-Gordon equation and by Orfanidis (1980) for the nonlinear Schrödinger and Korteweg–de Vries equations. As stressed by Neveu and Papanicolaou, the spectral parameter λ must be clearly distinguished from the Bäcklund parameter appearing in BT. In usual approaches, the two parameters are identified.

The gauge transformation (16) induces the corresponding BT for the associated sigma model:

$$\hat{Q} = \sum_{i=1}^{3} S_{3i} E_i \tag{17}$$

where S_{3i} are matrix elements of the three-dimensional representation of the matrix S entering into (16). Hence, for finding BT for the sigma model we must at first realize BT for the Ernst equation as a gauge transformation with the matrix $S \in SU(1,1)$ and obtain then three-dimensional representation of S.

BTs for the Ernst equation have been derived by Harrison (1978), Neugebauer (1979), Omote and Wadati (1981), Forgács et al. (1981), and Chinea (1983). We shall use the BT of Omote and Wadati which admits, however, considerable simplification. Namely,

$$\hat{M}_1 = -\frac{\theta-1+\kappa}{\theta-1-\kappa}M_1 - \frac{(\xi+\eta)^{-1}}{\theta-1-\kappa}, \qquad \hat{M}_2 = -\frac{\theta-1-\kappa}{\theta-1+\kappa}M_2 - \frac{(\xi+\eta)^{-1}}{\theta-1+\kappa}$$

$$\hat{N}_1 = -\frac{\theta+1+\kappa}{\theta+1-\kappa}N_1 + \frac{(\xi+\eta)^{-1}}{\theta+1-\kappa}, \qquad \hat{N}_2 = -\frac{\theta+1-\kappa}{\theta+1+\kappa}N_2 + \frac{(\xi+\eta)^{-1}}{\theta+1+\kappa} \tag{18}$$

Here $\theta = [i\tilde{c}/2(\xi+\eta)](\varepsilon+\varepsilon^*)(\hat{\varepsilon}+\hat{\varepsilon}^*)$, $\kappa^2 = \theta^2 + 1 + 2\theta(1+\zeta^2)(1-\zeta^2)^{-1}$, $\tilde{c}$ is arbitrary real parameter, $\zeta^2 = (il-\eta)(il+\xi)^{-1}$, l is a real Bäcklund parameter.

For finding the matrix S we shall use the two-dimensional representation of the $su(1,1)$ algebra. Let S be parameterized by

$$S = \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix}, \qquad aa^* - bb^* = 1 \tag{19}$$

The matrix elements a and b depend on ε, ε^*, $\hat{\varepsilon}$, $\hat{\varepsilon}^*$, ξ, and η. Taking into account the structure of BT (18) we assume all the dependence of a and b on $\varepsilon, \ldots, \hat{\varepsilon}^*$ is contained in single quantity θ, i.e., $a = a(\theta, \xi, \eta)$ and $b = b(\theta, \xi, \eta)$. We obtain two matrix differential equations

$$S_\xi = \hat{U}S - SU, \qquad S_\eta = \hat{V}S - SV$$

where U and V are given in (6) with two-dimensional realization of the generators $Z_\pm$ and Z_3, while $\hat{U}$ and $\hat{V}$ follow from U and V after substitutions $M_i \to \hat{M}_i$, $N_i \to \hat{N}_i$. Solving these equations yields in explicit terms

$$a = \frac{(\xi-i\lambda)(\theta+1+\kappa)+(\eta+i\lambda)(\theta-1+\kappa)}{2[i(\lambda-l)(\xi+\eta)\theta]^{1/2}},$$

$$b = \frac{1}{i}\left[\frac{(\xi-i\lambda)(\eta+i\lambda)}{i(\lambda-l)(\xi+\eta)\theta}\right]^{1/2} \tag{20}$$

The three-dimensional representation of S can be found in the framework of the extremely useful vector parametrization of the $SU(1,1)$ group (Bogush and Fedorov, 1977; Fedorov, 1978). We get (see Appendix for details)

$$S = I + \frac{(\xi+\eta)\kappa^2}{2i(\lambda-l)\theta}\begin{pmatrix} -\alpha^2 & \alpha\beta & -\alpha \\ \alpha\beta & -\beta^2 & \beta \\ \alpha & -\beta & -\alpha^2-\beta^2 \end{pmatrix} \tag{21}$$

where

$$\alpha = \frac{(\xi - i\lambda)(\theta + 1) + (\eta + i\lambda)(\theta - 1)}{i\kappa(\xi + \eta)}, \qquad \beta = \frac{2(\xi - i\lambda)^{1/2}(\eta + i\lambda)^{1/2}}{i\kappa(\xi + \eta)}$$

Inserting the matrix elements S_{3i} found explicitly into (17) we obtain BT for the sigma field Q:

$$\hat{Q} = \left[1 - \frac{(\xi + \eta)(\alpha^2 + \beta^2)\kappa^2}{2i(\lambda - l)\theta}\right]Q - \frac{(\xi + \eta)\kappa^2}{2i(\lambda - l)\theta}\left[(i\alpha - \beta)E_+ + (i\alpha + \beta)E_-\right]$$

5. DIVERGENCE-FREE CURRENTS

Knowledge of the explicit form for the connection coefficients U' and V' (14) allows us to derive an infinite number of divergence-free currents. We define a one-form ω by

$$\omega = \left(U'_{11} + \Gamma U'_{12}\right) d\xi + \left(V'_{11} + \Gamma V'_{12}\right) d\eta$$

where Γ is a solution of the Riccati equations

$$\Gamma_\xi = U'_{21} + \left(U'_{22} - U'_{11}\right)\Gamma - U'_{12}\Gamma^2, \qquad \Gamma_\eta = V'_{21} + \left(V'_{22} - V'_{11}\right)\Gamma - V'_{12}\Gamma^2$$

This form is closed, $d\omega = 0$, provided Q entering the matrix elements U'_{ij} and V'_{ij} is a solution of the sigma model equation. Identifying ω with a one-form $J_2\, d\xi - J_1\, d\eta$, where J_i is a corresponding parametric current we obtain parametric continuity equation $J_{1_\xi} + J_{2_\eta} = 0$. By expanding in power series of λ we get an infinite number of divergence-free currents $J_i^{(n)}$ $(n = 0, 1, \ldots)$. Their analytic form can be derived in the following manner.

Let us expand the sigma field Q in the $su(1,1)$ basis, $Q = \sum_{i=1}^3 q_i \tau_i$, where $-q_1^2 - q_2^2 + q_3^2 = 1$ and make use of a $su(1,1)$ analog of the stereographic projection:

$$q_1 = i\frac{b^* - b}{1 - b^*b}, \qquad q_2 = -\frac{b^* + b}{1 - b^*b}, \qquad q_3 = \frac{1 + b^*b}{1 - b^*b}$$

where b is a complex function of ξ and η. Introducing then new fields $\mu = b + b^{*-1}$ and $\nu = b - b^{*-1}$ we can represent the matrix Q in the form

$$Q = -\frac{1}{2\nu}\begin{pmatrix} 2\mu & 4 \\ -(\mu + \nu)(\mu - \nu) & -2\mu \end{pmatrix}$$

Subsequent calculation is straightforward.

We give here the first two currents

$$J_1^{(0)} = \nu_\eta \nu^{-1}, \qquad J_2^{(0)} = \nu_\xi \nu^{-1} \tag{22}$$

$$J_1^{(1)} = -\frac{i}{4}\left(\frac{1}{\xi} + \frac{1}{\eta}\right) J_1^{(0)} - 2\mu_\eta s(\xi, \eta),$$

$$J_2^{(1)} = -\frac{i}{4}\left(\frac{1}{\xi} + \frac{1}{\eta}\right) J_2^{(0)} + 2\mu_\xi s(\xi, \eta) \tag{23}$$

Here $s(\xi, \eta)$ is a solution of a system of linear equations

$$s_\xi = -\frac{i}{8}\left(\frac{1}{\xi} + \frac{1}{\eta}\right)\frac{\mu_\xi}{\nu^2}, \qquad s_\eta = \frac{i}{8}\left(\frac{1}{\xi} + \frac{1}{\eta}\right)\frac{\mu_\eta}{\nu^2}$$

Equation (22) is the analog of the (local) isospin conservation law, while (23) and all the following ones are nonlocal divergence-free currents involving more and more integrations.

APPENDIX

Let us introduce a complex vector parameter $\mathbf{c} = \alpha\mathbf{e}_2 + \beta\mathbf{e} - \beta^*\mathbf{e}^*$, where $\mathbf{e}_2$, $\mathbf{e} = 2^{-1/2}(\mathbf{e}_1 - i\mathbf{e}_3)$ and $\mathbf{e}^* = 2^{-1/2}(\mathbf{e}_1 + i\mathbf{e}_3)$ are basis vectors, α and β are real and complex coordinates, respectively. Then any matrix from the $SU(1,1)$ group is represented as

$$\begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix} = \frac{1}{1 + \alpha^2 - |\beta|^2}\begin{pmatrix} 1 + i\alpha & \beta \\ \beta^* & 1 - i\alpha \end{pmatrix} \tag{A1}$$

Taking into account the explicit expressions for the matrix elements a and b (20) we find the following identification:

$$\alpha = \frac{(\xi - i\lambda)(\theta + 1) + (\eta + i\lambda)(\theta - 1)}{i(\xi + \eta)\kappa}, \qquad \beta = \frac{2(\xi - i\lambda)^{1/2}(\eta + i\lambda)^{1/2}}{i(\xi + \eta)\kappa} \tag{A2}$$

The three-dimensional representation of S is given by the formula

$$S(\mathbf{c}) = I + 2\frac{c^\times + c^{\times 2}}{1 + \mathbf{c}^2} \tag{A3}$$

where the elements of a 3×3 matrix $c^\times$ are related with components of the three-dimensional vector $\mathbf{c}$ by $(c^\times)_{ij} = \varepsilon_{ijk} c_k$, $i, j, k = 1,2,3$, ε_{ijk} is completely antisymmetric unit tensor, $\varepsilon_{123} = 1$. Taking then the vector $\mathbf{c}$ with coordinates α and β(A2) and constructing the corresponding matrix c^x (A3) yields the matrix S (21).

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